Some remarks concerning differential flatness and tangent systems

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Outline

1. Differential flatness - Geometric point of view

2. Tangent system, bases and integrability conditions

3. Systems with one input

4. Systems of codimension one

5. Conclusion
Differential flatness

Notation

- System
  \[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u^i \]
  \[ x \in \mathcal{X} \subset \mathbb{R}^n \]
  \[ u \in \mathbb{R}^m \]

- Assume \( \text{rank}(g_1(x) \ldots g_m(x)) = m \)
  we can (locally) eliminate the inputs:
  \[ F^i(x, \dot{x}) = 0, \quad i = 1, \ldots, n - m \]

- „Flat output“:
  \[ y = (y^i) \quad y^i = h^i(x, \dot{x}, \ldots) \quad i = 1, \ldots, m \]
  \[ x = (x^j) \quad x^j = k^j(y, \dot{y}, \ldots) \quad j = 1, \ldots, n \]
Differential flatness
Geometric point of view

Differential equations $F^i(x, \dot{x}) = 0$ define regular submanifold $S \subset J^1\pi$
Geometric point of view

Tangent system

Differential equation \( F(x, \dot{x}) = 0 \) defines regular submanifold \( S \subset J^1_\pi \)

Tangent space \( T_p S \) is defined by
\[
dF|_v = 0, \quad v \in T_p(J^1_\pi)
\]
\[
dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial \dot{x}} d\dot{x}
\]
Differential equation \( F(x, \dot{x}) = 0 \) defines regular submanifold \( S \subset J^1 \pi \)

All elements of \( T_p S \) are given by

\[
v = \sum_{k=1}^{m} \left( \sum_i v_{k,i} \frac{d^i}{dt^i} \right) \xi^k.
\]

"flat output of tangent system":

Dual basis \( \omega^i \in T^*_p J^1 \pi, \ i = 1, \ldots m \) such that \( \omega^i \mid v_k = \delta^i_k \)

Coordinates \( \xi^i \) are given by \( \xi^i = \omega^i \mid v = \omega^i(v) \)
Differential Flatness

Necessary and sufficient condition

Dual basis $\omega^i \in T^*_{p} J^1 \pi$, $i = 1, \ldots, m$ generates cotangent module $T^*_{p} S$.

Two bases are related by unimodular polynomial operator matrices

$$\tilde{\omega} = U \left( \frac{d}{dt} \right) \omega$$

$$U \left( \frac{d}{dt} \right) \in (\mathcal{K} \left[ \frac{d}{dt} \right])^{m \times m}$$

System $F(x, \dot{x}) = 0$ is flat

There is an integrable basis $\tilde{\omega}$ of $T^*_{p} S$

Integrability means $d\tilde{\omega}^i = 0$, $i = 1, \ldots, m$

Flat output of $F(x, \dot{x}) = 0$ is given by integration of $dh = \tilde{\omega}$
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Systems with with one input

**Tangent system**

System

\[ \dot{x} = f(x) + g(x)u \quad x(t) \in \mathcal{X} \subseteq \mathbb{R}^n, u(t) \in \mathbb{R} \]

Linearization

\[ d\dot{x} = \left( \frac{\partial f}{\partial x}(\sigma) + \frac{\partial g}{\partial x}(\sigma)\sigma_u \right) dx + g(\sigma)du \]

\[ \begin{bmatrix} A & b \\ \end{bmatrix} \]

at solution

\[ \sigma : t \mapsto x(t) \]

\[ \sigma_u : t \mapsto u(t) \]

Controllability matrix of tangent system

\[
P = \begin{pmatrix} b & (A - \frac{d}{dt})b & \cdots & (A - \frac{d}{dt})^{n-1}b \end{pmatrix}
\]

\[
= \begin{pmatrix} g & \text{ad}_{-(f+gu)}g & \cdots & \left( \text{ad}_{-(f+gu)} - \sum_i \frac{\partial}{\partial u^{(i)}} g \right)^{n-1} g \end{pmatrix}_{(\sigma,\sigma_u)}
\]
Systems with with one input

Tangent system and Integrability condition

- Flat output of tangent system
  \[ \omega = q dx = \sum_{i} q_i dx^i \]
  is given by the last row of inverse controllability matrix
  \[ q \left( g \quad \text{ad}_{-f+gu} g \quad \cdots \quad \left( \text{ad}_{-f+gu} - \sum_i u^{(i)} \frac{\partial}{\partial u^{(i)}} \right)^{n-1} g \right) = (0 \quad \cdots \quad 0 \quad 1) \]
- System \( \dot{x} = f(x) + g(x)u \) is flat, if and only if
  \[ d\omega \wedge \omega = 0 \]
- in this case \( \omega \) also annihilates distribution spanned by
  \[ \left( g \quad \text{ad}_{-f} g \quad \cdots \quad \text{ad}_{-f}^{n-2} g \right) \]
  \[ -> \text{dual version of well known flatness condition for } m = 1 \]
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Systems with n states and n-1 inputs

Tangent system

\[ \dot{x} = f(x) + \sum_{i=1}^{n-1} g_i(x)u^i \quad x(t) \in X \subset \mathbb{R}^n, \quad u^i(t) \in \mathbb{R} \]

Linearization

\[ d\dot{x} = \left( \frac{\partial f}{\partial x}(\sigma) + \sum_{i=1}^{n-1} u^i(t) \frac{\partial g_i}{\partial x}(x(t)) \right) dx^j + \sum_{i=1}^{n-1} g_i(x(t))du^i \]

\[ A(t) \]

Assume controllability matrix of tangent system has full rank

\[ (b_1 \quad \cdots \quad b_{n-1} \quad (A - \frac{d}{dt})b_1 \quad \cdots \quad (A - \frac{d}{dt})b_{n-1}) \]

Reorder inputs, such that

\[ P = (g_1 \quad \cdots \quad g_{n-1} \quad \text{ad}_-(f+\sum_i g_iu^i)g_{n-1}) \]

has also full rank.
Systems with with n states and n-1 inputs

Tangent system

- Flat output of tangent system is given by
  \[
  \omega = \begin{pmatrix}
  \omega^1 \\
  \vdots \\
  \omega^{n-2} \\
  \omega^{n-1}
  \end{pmatrix}
  = \begin{pmatrix}
  g_1^T(x) \\
  \vdots \\
  g_{n-2}^T(x) \\
  g_{n-1}(x)
  \end{pmatrix}
  \ dx
  \]

- Integrability condition (Frobenius):
  \[
  g^\perp(x)g_i(x) = 0
  \]
  Dimension!
  \[
  d\omega^i \wedge \omega^1 \wedge \ldots \wedge \omega^{n-1} = 0, \quad i = 1, \ldots, n - 1
  \]

- Hence, there is a matrix \( \mu_i^j(x) \) (integrating factor), such that
  \[
  dh^j = \sum_{i=1}^{n-1} \mu_i^j(x)\omega^i
  \]

- Integration, gives flat output \( y^j = h^j(x) \) of nonlinear system
Conclusion

- Short view on geometry of differentially flat systems
- Flatness conditions in terms of integrability condition of flat outputs of the tangent system
- Dual version of the well-known condition for systems with one input
- Easier proof of the sufficient condition for systems with n states and n-1 inputs

Thank you!

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